International J. of Multidispl. Research & Advcs. in Engg. (IJMRAE),
ISSN 0975-7074, Vol. 9 No. I (April, 2017), pp. 65-71

ON ENDOMOMORPHISMS AND ANTIENDOMOMORPHISMS OF LIE IDEALS IN PRIME RINGS WITH INVOLUTIONS

REKHA RANI
Department of Mathematics,
S. V. College Aligarh, India
E-mail: rekharani@rediffmail.com

Abstract

Let R be a $*$-prime ring *char* different from 2 and L be a square closed $*$-Lie ideal of R. Suppose that R admits a generalized left $*$-derivation $F : R \to R$. If F acts as a homomorphism or as an anti-homomorphism on L, then F is right $*$-centralizer on R.

Key Words and Phrases : *Generalized left $*$-derivations, Jordan $*$-centralizer, Prime rings, Lie ideals.*

2000 AMS Subject Classification : 16N60, 16W25, 16W10, 16U80.

1. Introduction

Throughout this paper, R will denote an associative ring (may be without unity 1, unless specifically their use) with center $Z(R)$. The ring R is n-torsion free for any prime integer where $n > 1$ is an integer, if $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall frequently use the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution on R if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with an involution or $*$-ring. A ring with an involution '$*$' is said to be a $*$-prime ring if $aRb = aRb^* = \{0\}$ (or $aRb = a^*Rb = \{0\}$) implies either $a = 0$ or $b = 0$. It is obvious that every prime ring with an involution '$*$' is a $*$-prime ring but the converse may not be necessarily true in general. An additive subgroup L of R is said to be Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L is said to be square closed if $u^2 \in L$ for all $u \in L$ and L is said to be $*$-Lie ideal if $L = L^*$.

An additive mapping δ is called derivation (resp. Jordan derivation) if $\delta(xy) = \delta(x)y + x\delta(y)$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$) holds for all $x, y \in R$. An additive mapping $f : R \to R$ is called generalized derivation if there exists a derivation $\delta : R \to R$ such that $f(xy) = f(x)y + x\delta(y)$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ is said to be left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in R$. Clearly, every left derivation on a ring R is a Jordan left derivation but the converse need not be true in general Z)(see for example 16, 1.1].

Let S be a nonempty subset of R and $\delta : R \to R$ be a derivation of R. If $\delta(xy) = \delta(x)\delta(y)$ (resp. $\delta(xy) = \delta(y)\delta(x)$) for all $x, y \in S$, then δ is said to acts as homomorphism (resp. anti-homomorphism) on S. In [4], Bell and Kappe proved that if I is a nonzero right ideal of a prime ring R and δ is a derivation of R such that δ acts as homomorphism (resp. anti-homomorphism) on I, then $\delta = 0$. Further, this result was extended by Rehman [12] for generalized derivation. Similar types of results have been proved for generalized left derivation by Rehman and Ansari in [15].

2. Generalized Left $*$-derivation Motivated by the definitions of $*$-derivation and generalized $*$-derivation, Rehman and Ansari [15] recently introduced the notions of left $*$-derivation and generalized left $*$-derivation as follows: Let R be a $*$-ring. An additive mapping $d : R \to R$ is said to be left $*$-derivation if $d(xy) = x^*d(y) + yd(x)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be generalized left $*$-derivation if there exists a left $*$-derivation d such that $F(xy) = x^*F(y) + yd(x)$ for all $x, y \in R$. The concept of generalized left $*$-derivations cover the concept of left $*$-derivations. Moreover, a generalized left $*$-derivation with $d = 0$ includes the concept of right $*$-centralizer i.e., an additive mapping $T : R \to R$ satisfying $T(xy) = x^*T(y)$ for all $x, y \in R$. In [4], Bell and Kappe discussed the derivations acting as a homomorphisms or an anti-homomorphisms on a nonzero right ideal of a prime ring. Recently, Shakir [1], proved some results taking generalized left derivation of a prime ring R which acts either as homomorphisms or as an anti-homomorphisms on a certain well behaved subset of R. Further, this result was extended by Rehman and Ansari in [15] in the setting of generalized left $*$-derivation and generalized left $*$-bi-derivation. In the present section our objective is to extend the results obtained for Lie ideals.
In order to prove our main results, we need the following Lemmas:

Lemma 2.1[5, Lemma 4]. Let R be a $*$-prime ring of characteristic different from two, L be a nonzero $*$-Lie ideal of R and $a, b \in R$. If $aLb^* = \{0\}$ then $a = 0$ or $b = 0$ or $L \subseteq Z(R)$.

Lemma 2.2[7, Lemma 3.3]. Let R be a $*$-prime ring of characteristic different from two, L be a nonzero $*$-Lie ideal of R. If $a \in R$ such that $[a, L] \subseteq Z(R)$, then either $a \in Z(R)$ or $L \subseteq Z(R)$.

Lemma 2.3[10, Lemma 2.3]. Let R be a $*$-prime ring of characteristic different from two, L be a nonzero $*$-Lie ideal of R. Suppose $[L, L] \subseteq Z(R)$, then $L \subseteq Z(R)$.

Let R be a $*$-prime ring of $charR \neq 2$ and $L \nsubseteq Z(R)$ is a square closed $*$-Lie ideal of R with involution '$*$'. Suppose that $F : R \to R$ is a generalized left $*$-derivation with associated left $*$-derivation on R. If F acts as a homomorphisms on L, then F is right $*$-centralizer on R.

Proof. If F acts as a homomorphisms on L, then $F(xy) = F(x)F(y)$ for all $x, y \in L$ and also from the definition of generalized left $*$-derivation, we have $F(xy) = x^*F(y) +$

$yd(x)$ for all $x, y \in R$, where d is a left $*$-derivation of R. Now

$$F(xyz) = F(x(yz)) = x^*F(yz) + yzd(x). \tag{2.1}$$

$$= x^*F(y)F(z) + yzd(x) \text{ for all } x, y, z \in L.$$

On the other hand

$$F(xyz) = F((xy)z) = F(xy)F(z)$$

$$= x^*F(y)F(z) + yd(x)F(z) \text{ for all } x, y, z \in L.$$

Combining (2.1) and (2.2), we obtain $x^*F(y)F(z)+yzd(x) = x^*F(y)F(z)+yd(x)F(z)$, for all $x, y, z \in L$. This yields that $y\big(zd(x) - d(x)F(z)\big) = 0$ for all $x, y, z \in L$. Multiplying left side by $zd(x) - d(x)F(z)$ to the above relation, we obtain $\big(zd(x) - d(x)F(z)\big)y\big(zd(x) - d(x)F(z)\big) = 0$ for all $x, y, z \in L$. Then by Lemma 2.1, we obtain

$$zd(x) - d(x)F(z) = 0 \text{ for all } x, z \in L. \tag{2.3}$$

Replacing x by $2xy$ and using the fact that $char R \neq 2$ in the above relation, we get

$$zx^*d(y) + zyd(x) - x^*d(y)F(z) - yd(x)F(z) = 0 \text{ for all } x, y, z \in L.$$

Using relation (2.3) in the above relation, we find that

$$zx^*d(y) + zyd(x) - x^*zd(y) - yzd(x) = 0 \text{ for all } x, y, z \in L.$$

This yields that

$$[z, x^*]d(y) + [z, y]d(x) = 0 \text{ for all } x, y, z \in L.$$

In particular, replacing y by z in the above relation, we find that

$$[z, x^*]d(z) = 0 \text{ for all } x, z \in L.$$

Again, replace x by x^* in the above expression, to obtain

$$[x, z]d(z) = 0 \text{ for all } x, z \in L. \tag{2.4}$$

Replacing x by $2xy$ and using the fact that $char R \neq 2$, we get $[x, z]yd(z) = 0$, for all $x, y, z \in L$. Now, multiplying left side by $d(z)$ and right side by $[x, z]$, and by Lemma

2.1, we get $d(z)[x,z]=0$ for all $x,z\in L$. Then, replacing z by y and linearizing the above relation, we obtain $d(x)[z,y]+d(z)[x,y]=0$ for all $x,y,z\in L$ and hence

$$d(x)[z,y]=-d(z)[x,y] \text{ for all } x,y,z\in L. \tag{2.5}$$

Replacing y by $2uy$ in (‘2.5) and again using (2.5), we get $2d(x)u[x,y]=0$ for all $x,y,u\in L$. Since $char R\neq 2$, we find that $d(x)u[x,y]=0$. Now, replace u by $2[z,y]r$ and use the fact that $char R\neq 2$, to get $d(x)[z,y]r[x,y]=0$ for all $x,y,z\in L$ and $r\in R$ and hence application of (2.5), we obtain $d(z)[x,y]r[x,y]=0$ for all $x,y,z\in L$ and $r\in R$. Again replacing r by $rd(z)$ in the above expression, we get $d(z)[x,y]rd(z)[x,y]=0$ for all $x,y,z\in L$ and $r\in R$, that is, $d(z)[x,y]Rd(z)[x,y]=\{0\}$ for all $x,y,z\in L$. Thus primeness of R forces that $d(z)[x,y]=0$ for all $x,y,z\in L$. Again, replacing x by $2tx$ and using the fact that $char R\neq 2$, we get $d(z)t[x,y]=0$ for all $x,y,z,t\in L$. Again by Lemma 2.2, we get $d(z)=0$ for all $z\in L$. Replacing z by $2r[y,z]$ and using $char R\neq 2$, we obtain $[y,z]d(r)=0$ for all $y,z\in L$ and $r\in R$. Again, replacing y by $2yx$ and using $char R\neq 2$, we get $[y,z]xd(r)=0$ for all $x,y,z\in L$ and $r\in R$. Therefore, by Lemma2.2, we get $d=0$ on R. Therefore, $F(xy)=x^*F(y)$ for all $x,y\in R$. Hence, we get the required result.

Let R be a $*$-prime ring of $char R\neq 2$ and $L\nsubseteq Z(R)$ is a square closed $*$-Lie ideal of R with involution ‘$*$’. Suppose that $F:R\to R$ is a generalized left $*$-derivation with associated left $*$-derivation on R. If F acts as an anti-homomorphisms on L, then F is right $*$-centralizer on R.

Proof. If F acts as an anti-homomorphisms on L, then $F(xy)=F(y)F(x)$, for all $x,y\in L$. By the definition of generalized left $*$-derivation, we have

$$F(xy)=x^*F(y)+yd(x), \text{ for all } x,y\in L. \tag{2.6}$$

Replacing y by $2xy$ in (2.6) and using the fact that $char R\neq 2$, we get

$$\begin{aligned} F(x(xy)) &= x^*F(xy)+xyd(x)\\ &= x^*\{x^*F(y)+yd(x)\}+xyd(x)\\ &= x^*x^*F(y)+x^*yd(x)+xyd(x), \text{ for all } x,y\in L. \end{aligned} \tag{2.7}$$

On the other hand

$$F((xx)y) = (xx)^*F(y) + yd(xx)$$
$$= x^*x^*F(y) + 2yxd(x), \text{ for all } x, y, z \in L. \quad (2.8)$$

Equating (2.7) and (2.8), we get $x^*x^*F(y)+x^*yd(x)+xyd(x) = x^*x^*F(y)+2yxd(x)$, for all $x, y, z \in L$. This yields that

$$(x^*y + xy)d(x) = 2yxd(x), \text{ for all } x, y \in L. \quad (2.9)$$

Replacing y by $2yz$ in (2.9) and using the fact that $char R \neq 2$, we get $(x^*y+xy)zd(x) = 2yzxd(x)$, for all $x, y, z \in L$. Using (2.9), this relation gives that $(x^*y + xy)zd(x) = y(x^*z + xz)d(x)$, which yields that $[x^*, y]zd(x) + [x, y]zd(x) = 0$, for all $x, y, z \in L$. Now replace x^* by x, to get $2[x, y]zd(x) = 0$, for all $x, y, z \in L$. Since $Char R \neq 2$, so we have $[x, y]zd(x) = 0$, for all $x, y, z \in L$. By Lemma 2.1, this relation implies that either $[x, y] = 0$ or $d(x) = 0$. Since $L \neq Z(R)$, so we have $d(x) = 0$, for all $x \in L$. Now the result follows the proof of Theorem 2.1.

We immediately get the following corollary from the above theorems: Let R be a $*$-prime ring of $char R \neq 2$ with involution '$*$'. Suppose that $d : R \to R$ is a left $*$-derivation on R. If d acts as a homomorphisms or anti-homomorphisms on R, then d is right $*$-centralizer on R.

References

[1] S. Ali, On generalized left derivations in rings and Banach algebras, Aequ. Math. 81 (2011), 209-226.

[2] M. Ashraf and Khan, A., Commutativity of $*$- prime rings with generalized derivations, Rend. Se. Mat. Univ. Padova, 125, 71-79(2011).

[3] M. Ashraf and N. Rehman, On Lie ideals and Jordan left derivations of prime rings, Arch. Math. (Brno) 36, (2000), 201-206.

[4] H. E. Bell and L. E. Kappe, Ring in which derivations satisfying certain algebraic conditions, Acta Math. Hungar. 53 (1989), 339-340.

[5] Bergen, J., Herstein, I.N., Kerr, J.W., *Lie ideals and derivations of prime rings*, J. Algebra 71 (1981), 259-267.

[6] I. N. Herstein, Topics in ring theory, University of Chicago Press, Chicago, London, 1969.

[7] Herstein, I. N., *A note on derivations*, Canad. Math. Bull. 21 (1978), 369-370.

[8] Gy. Maska, Remark on symmetric biadditive functions having non-negative diagonalization, Glasnik Matematicki 15, (1980), 279-280.

[9] N. M. Muthana, Left centralizer traces, generalized biderivations, left bi-multipliers and generalized Jordan biderivations, Aligarh Bull. Math. 26(2)(2007), 33-45.

[10] Oukhtite, L. and Salhi, S., Lie ideals and derivations of $*$- prime rings, Int. J. Algebra 1(2007), 25-30.

[11] Oukhtite, L. and Salhi, S., Centralizing Automorphisms and Jordan left derivations on $*$- prime rings, Adv. Algebra 1(1)(2008), 19-26.

[12] Rehman, N., On generalized derivation as homomorphisms and anti-homomorphisms, Glasnic Mat. 39(59) (2004),27-30.

[13] N. Rehman and A. Z. Ansari, On Lie ideals and generalized Jordan left derivations of prime rings, Ukrainian J. Mathematics 65 (8), (2013) 1118-1125.

[14] N. Rehman, A. Z. Ansari and C. Heitenger, A note on homomorphism in rings with involution, Thai J. Mathematics, Volume 11 (2013) Number 3 741-750.

[15] N. Rehman and A. Z. Ansari, Generalized left derivations acting as homomorphisms and anti- homomorphisms on Lie ideal of rings, accepted for publication in Journal of Egyptian Math. Society.